Two Dimensions Gravity Inverse Problem Using Adaptive pruning L-Curve Technique

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Abstract

This paper applies the truncated singular value decomposition method (TSVD) with adaptive pruning L-curve technique to solve numerically the two dimensions ill-posed inverse problem of gravity anomalies. The inverse problem here is to determine a plausible spatial variation of density within the earth that is consistent with a finite set of geophysical observations. To reduce the ambiguity; the problem was solved as an overdetermined one and linear constraints were added. The results obtained are compared to exact solutions of simple and complicated synthetic earth models with and without noise. It is found that TSVD with adaptive pruning L-curve technique is stable, robust and it always converges to the right solutions. Furthermore, the method is successfully applied to a real data example from USA.

Keywords: Gravity inversion, truncated singular value decomposition, L-curve, pruning.

Introduction :

The main target in gravity as in other geophysical prospecting methods is to find a plausible causative subsurface body from surface observation. Many authors divide the earth into an array of rectangular prisms with predefined geometry and the unknowns are the density (or any other physical property) of each prism, which is supposed to be constant and homogenous (e.g., [18]; [19]; [20]; [24], [27]; [28]; [29] and many others). Such problems are expressed as a linear system of equations.

The numerical treatment of any linear system of equations (Ax=b) is not always easy. One has to check how large the condition number of matrix A. Numerical tools such as the singular value decomposition (SVD), can identify the linear dependencies and thus help to improve the model and lead to a modified system with a better-conditioned matrix. This modified system can then be solved by standard numerical techniques [4]. [13] postulated that any discussion of ill-conditioned matrices requires knowledge of the SVD of the matrix A. In particular, the condition number of A is defined as the ratio between the largest and the smallest singular values of A. Hansen [15] classified ill-conditioned system into two main classes. First, Rank-deficient problems, which are characterized by the matrix A having a cluster of small singular values, and there is a well-determined gap between large and small singular values. Such problems can be solved by extracting the linearly independent information in A, to arrive to another problem with a well-conditioned matrix. The second class of problems named, discrete ill-conditioned problems. In such problems, the singular values components of the solution, on the average, decay gradually to zero.

The inverse problem considered in this paper is to determine a plausible spatial variation of density within the earth that is consistent with a finite set of geophysical observations. This problem, due to the above explanation, can be classified as a discrete ill-conditioned problem. The regularization of a discrete ill-posed problem is a matter of finding out which erroneous SVD components to filter out [15]. It is more than merely filtering out a cluster of small singular values. There are many mathematical tools to apply like filter factors [9], the resolution matrix [30], L-curve [23] … etc. The TSVD method with adaptive L-curve technique is presented to solve the two dimensions ill-posed inverse problem of gravity anomalies. The advantage of this technique is to overcome the difficulties arising when we solve the noisy ill-posed system by using the standard SVD method.

Formulation of the Problem:

Let us assume that the domain of the subsurface model can be divided into two dimensions arrays of $\boldsymbol{N} \times \boldsymbol{M}$ rectangular equidimensional prisms, each prism has a constant density, as in Figure 1. Assume also that each prism has a uniform density $\boldsymbol{v}_i$, $i = 1,...,N$. Then, the gravity effect of the i-th data point is given by [19]:

$$g_i=a_{ij}d_j+e_i, \qquad i=1, \ldots, n, j=1, \ldots m \tag{1}$$

where, d_i is the density of the j^{th} block, $\boldsymbol{e}_i$ the noise associated with i^{th} data point, and $\boldsymbol{a}_{ij}$ is the matrix element representing the influence of the j^{th} block on the i^{th} gravity value. In the matrix notation equation **(1)** can be expressed as:

$$\boldsymbol{G}=\boldsymbol{AD}+\boldsymbol{E} \tag{2}$$

where, G is the gravity effect, $\boldsymbol{A}$ is the data kernels, $\boldsymbol{D}$ is the densities of the prisms, and $\boldsymbol{E}$ is the noise. The gravity inversion problem is the following: given $\boldsymbol{G}$, the observed gravity data, find a density distribution $\boldsymbol{D}$ which explains $\boldsymbol{G}$, taking into account a certain noise level. It is well known that the gravity inversion problem is an ill-posed problem, and characterized by unstable solutions [19].

TSVD method:

Let us consider the linear system (1), where, $\mathbf{A} \in \quad {}^{\times}$, $m \geq n$ is a full-rank matrix. As we know the least-squares solution can be expressed in the form:

$$d_{Ls} = \sum_{i=1}^{n} \frac{u_i^T g}{\sigma_i} v_i, \tag{3}$$

If the matrix A is rank-deficient, i.e., the singular value $\sigma_{\min} = 0$. It's easy to solve (1) by ignoring the SVD components associated with the zero singular value. However, there is one or more small but nonzero singular values so we say that A is numerically rank-deficient, i.e., there exists an integer $k = r_\varepsilon(\mathbf{A}) \leq n$ such that, for a given tolerance ε

$$\sigma_l \leq \varepsilon, \text{ for all } l = k+1, \ldots, n. \tag{4}$$

This integer k is, in fact, the numerical ε-rank of A, usually defined as

$$r_\varepsilon(\boldsymbol{A}) := \min_{\|E\|_2 \leq \varepsilon} rank(\boldsymbol{A} + \boldsymbol{E}), \tag{5}$$

Hansen in (15, page 49) reported that the most common regularization strategy for numerical rank-deficient problems consists of two steps

1. Replace the matrix $\boldsymbol{A}$ by a matrix of rank k. The usual choice for this rank-k matrix is:

$$A_k = \sum_{i=1}^{k} u_i \, u\sigma_i v_i^T$$

Where, A_k is the closest rank-k matrix to A .i.e., we replace the small singular values σ_l by zeros, $l = k+1,...,n.$

2. Compute the approximation solution by

$$d_k = A^+ g = \sum_{i=1}^{k} \frac{u_i^T g}{\sigma_i} v_i ,$$

This regularization strategy is known as TSVD and the parameter k is called truncation parameter (see [15]).

Ambiguity control:

Potential-field interpretation is characterized by an inherent ambiguity in the determination of the source from field data, which may leads to a loss of depth information [10]. Blakely [2] mentioned that by Green's third identity, any potential field in a subregion can be reproduced by an infinite variety of surface (shallow) distributions. Also, the annihilator or the source distribution which produces a null field couldn't be determined [26]. If the system is underdetermined, i.e. with more unknowns than data, this leads to algebraic ambiguity. Many authors solved such ambiguity problems by using additional information about the problem. The simplest way is to use a parametric discretization [22], in which the solution is assumed to consist of basic geometric body shapes with homogeneous source distributions. So, using 2-D array of prisms or using the model as a set of vertical prisms of variable depth to top solves such algebraic ambiguity. The field derivatives strongly contain the position of the prism source's boundaries [3]; [8]; [12]. Other additional a priori information is implementing lower and upper density bounds and for a density monotonically increasing with depth [11], imposing a condition of minimum volume to the causative body [19], constraining the source to have minimum momentum of inertia [14], requiring compactness along several axes using a priori information about the axes' length [1], using approximate equality (linear) constraints [21] and many other ideas to reduce the ambiguity.

The ambiguity is controlled in the present paper by first, using the 2-D array of prisms to fix the geometry of the model, assuming homogeneous density for each prism. Second, the number of data points is set to be larger than the unknown parameters (overdetermined problem). Finally, linear constraints about the densities of the surface prisms (or others if information is available from wells) are added.

L-curve regularization method:

The L-curve analysis plays an important role in the analysis phase of regularization problems. It is a plot for all valid regularization parameters of the discrete smoothing norm $\Omega(x_{reg})$ - e.g, the (semi) norm $\|Lx_{reg}\|_2$- of the regularized solution versus the corresponding residual norm $\|Ax_{reg} - b\|_2$. The L-curve clearly displays the compromise between minimization of these two quantities, which is the heart of any regularization method [15]. The optimal regularization parameter can be determined by locating the corner of such L-curve.

There are many methods to locate the corner e.g. using spline curve [17] and a more recent method called triangle method [5]. As the spline curve has an undesired tendency to track the unwanted local corners of the discrete L-curve, and therefore a preprocessing stage of smoothing was added. As the smoothing step depends on few parameters, the method is not adapted, [16]. They also stated that the triangle method is complex and the increase in the number of parameters increases its complexity and time taken to calculate. A new method, named, adaptive pruning method has proved great efficiency in determining the corner more than the other two mentioned methods.

The Adaptive Pruning Algorithm:

Hansen et al. [17] describe a robust and adaptive implementation of the L-curve criterion. The algorithm locates the corner of a discrete L-curve consisting of a log-log plot of corresponding residual and solution norms of regularized solutions from a method with a discrete regularization parameter (such as the TSVD used here).To achieve the required adaptivity and robustness, the new algorithm consists of two stages. In the first stage we compute corner candidates using L-curves at different scales or resolutions (not knowing a priori which scale is optimal). In the second stage we then compute the overall best corner from the candidates found in the first stage.

Algorithm:

1- *Input m and n (number of parameters and data points).*
2- *Compute **A** (data kernel matrix).*
3- *If (synthetic), calculate b and add noise (o, 90, or 80 dB)*
   Else if (real) read b from file.
4- *Input the number of constraints and the values them.*
5- *Update **A** and **b** with such constraints.*
6- *Compute **[U, s, V] = csvd(A)***
7- *Calculate rho (residual norm) and eta (solution norm).*
   $\|g - Ad_i\|, \ \|d_i\|, \quad \mathrm{i} = 1,\ldots,\mathrm{p}$
8- *Use adaptive pruning algorithm to locate the corner and draw the L-curve.*
9- *Calculate TSVD solution for the truncation level determined.*
10- *Convert the proper regularized solution vector 'd' into two dimensional array to draw the earth model as an image.*
11- *If synthetic then draw the solution and the true earth models*
   Else if real then draw solution earth model only.

12- *Draw observed and inverted gravity fields*

Our code is all written with Matlab 7.1, Calculating the forward (for synthetic examples) and inverse models, gravity effect, and drawing the earth models together with observed and inverted gravity fields. The regularization routines (Hansen, 2001) are included as functions. The code is automized to accept data, scale and linear constraints to start inversion and return back all drawings and results.

Application to Synthetic Models

The regularized TSVD method (using L-curve together with pruning algorithm) was tested on several synthetic earth models. We present here, as an example, the results obtained for three different earth models. Model 1 consists of 25 columns and 6 rows, i.e. 150 blocks to invert for. The dimension of each block is 2m × 2m. The number of data points is 150 points, i.e. evenly-determined problem. The model includes an inclined dyke of density of 0.3gm/cm^3 and a vertical one of density 0.25gm/cm^3. Term density in all models is referred to density contrast.

The result is present in the form of true (synthetic) and inverted gravity fields together with the true and inverted density models shown in Figure (2). Figure (3) shows the L-curve and the corner determined by adaptive pruning algorithm. The corner is at 141, which means only 9 rows were cut. So, relatively few information was lost. The data were then contaminated by 90 DB noise. Figure (4) shows the regularized solution, where the L-curve corner is at 58. This means higher loss in information. Figures (5) shows the results of Model 1 contaminated with 80DB noise. Model 2 consists of 8×18 prisms each of 1m×1m. So, the unknown parameters are 144. The corner of the L-curve is at 97 without adding any noise (Figure 6). The inverted model shows good agreement with the true one (Figure 7). Results of Model 2 after adding 90DB noise are displayed in Figure (8). Also higher noise were added (80DB) and the results are displayed in Figure (9).

Application to field examples:

To illustrate the applicability of the proposed method; it was applied to an isolated Bouguer anomaly over San Jacino graben, southern California, USA. The sediments filling the graben display increasing density with depth are reported by [7]. The earth model was divided into 6×20 prisms of dimension 0.5 km×0.5 km. The number of stations was increased to 240 by linear interpolation using Matlab, while the unknowns were only 120. The equality constraints of the surface prisms were added. The resulted L-curve (Figure 11) shows that the proper solution is at the truncation level 62. The solution is displayed in Figure (12). It is clear that the density contrast decreases, i.e. density increases with depth proving compatibility with previous inversion results obtained by [7], [25], [6] and [21].

Conclusion:

We present a gravity inversion problem solved by truncated singular value decomposition, of proper truncation level using adaptive pruning L-curve. The formulation of the problem as an overdetermined one by increasing the data points helps together with the linear constraints to get a plausible solution. The proposed method may be applied to variety of geologic problems, including dykes, basins, geologic contacts, tunnels and salt domes. The residual or separated anomaly can be used for direct inversion.

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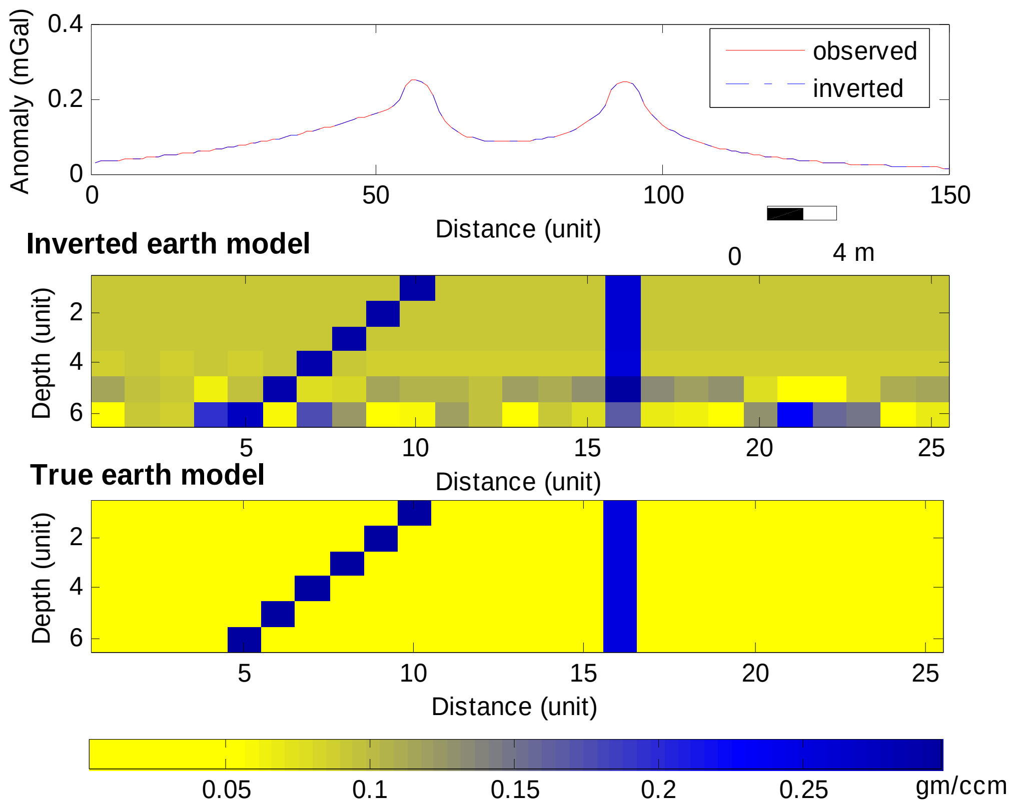

Figure 2. Inversion results for model 1 without adding any noise to the observed (calculated) gravity field.

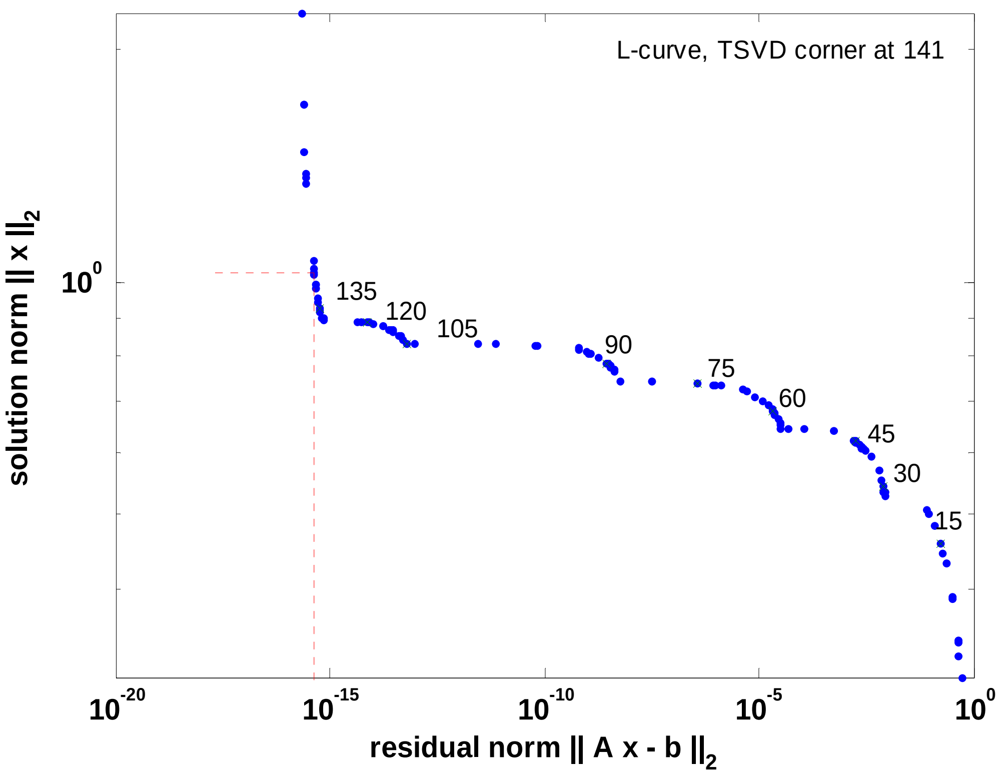

Figure 3. L-curve for Model 1 (the observed is free of noise).

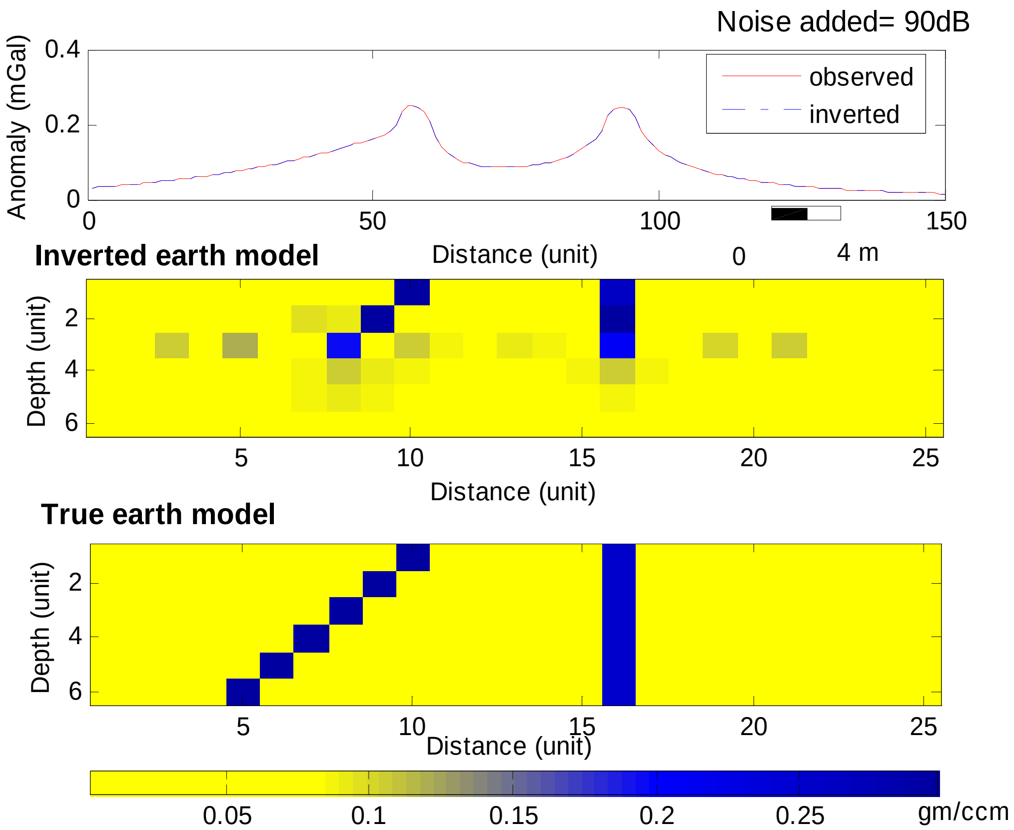

Figure 4. Inversion results of Model 1. After adding 90DB to the observed (calculated) gravity field.

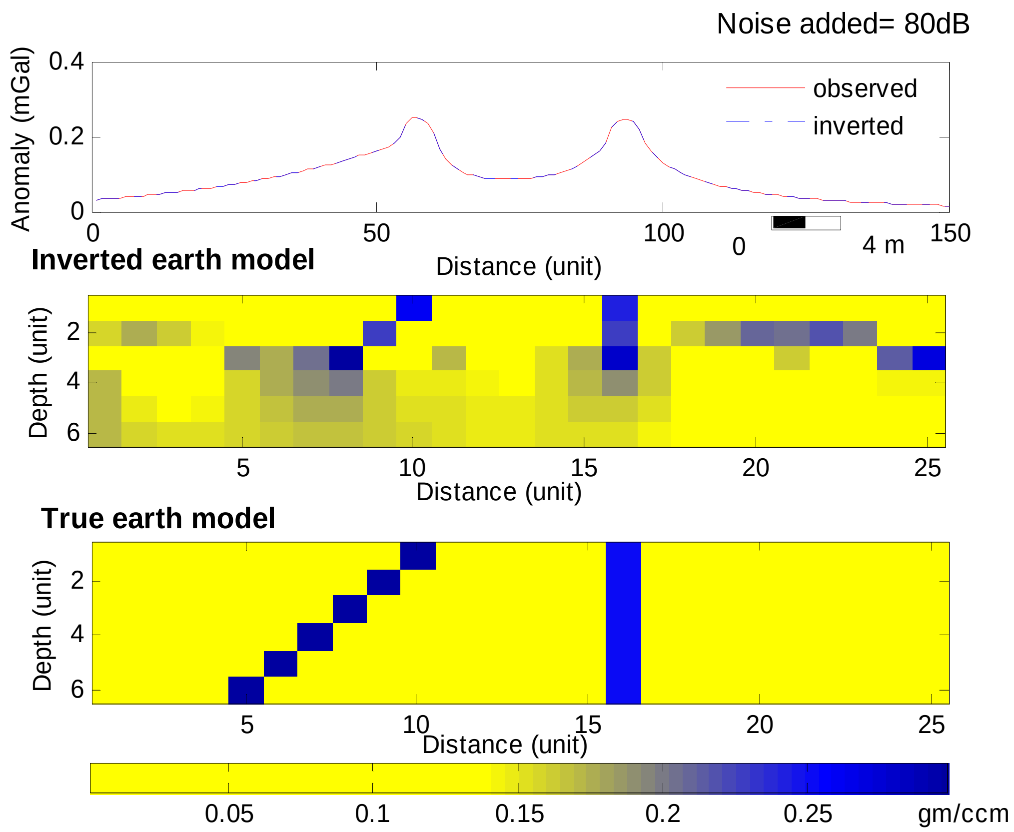

Figure 5. Inversion results of Model 1 after adding 80 DB noise.

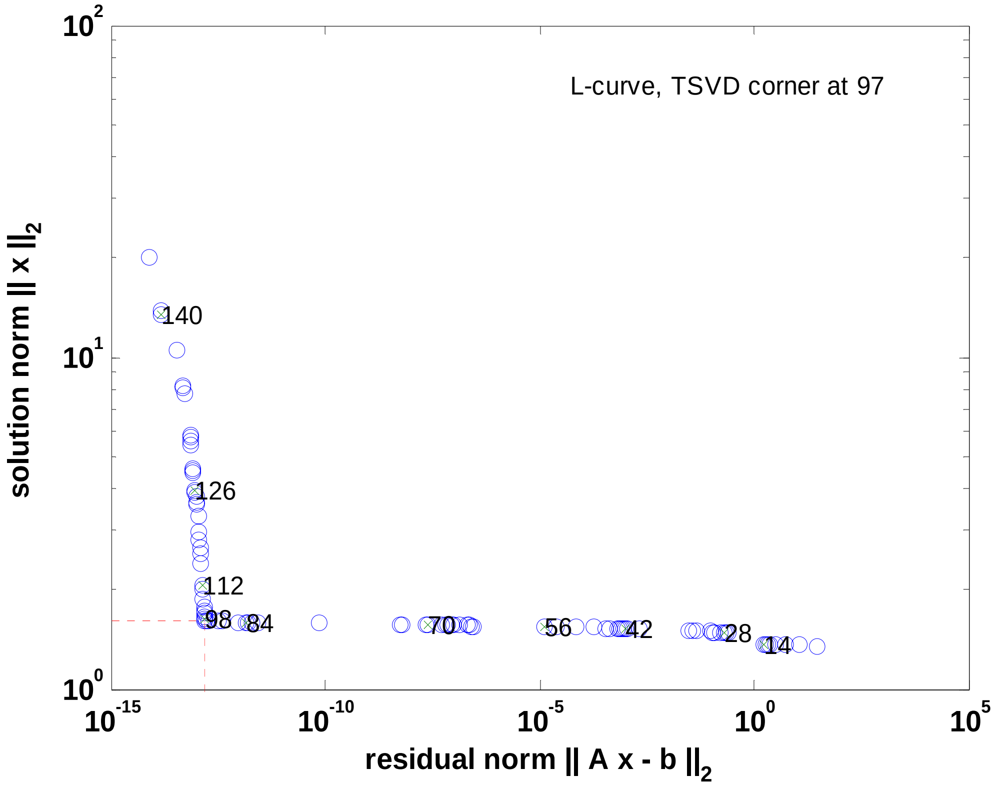

Figure 6. L-curve for Model 2 (the observed is free of noise).

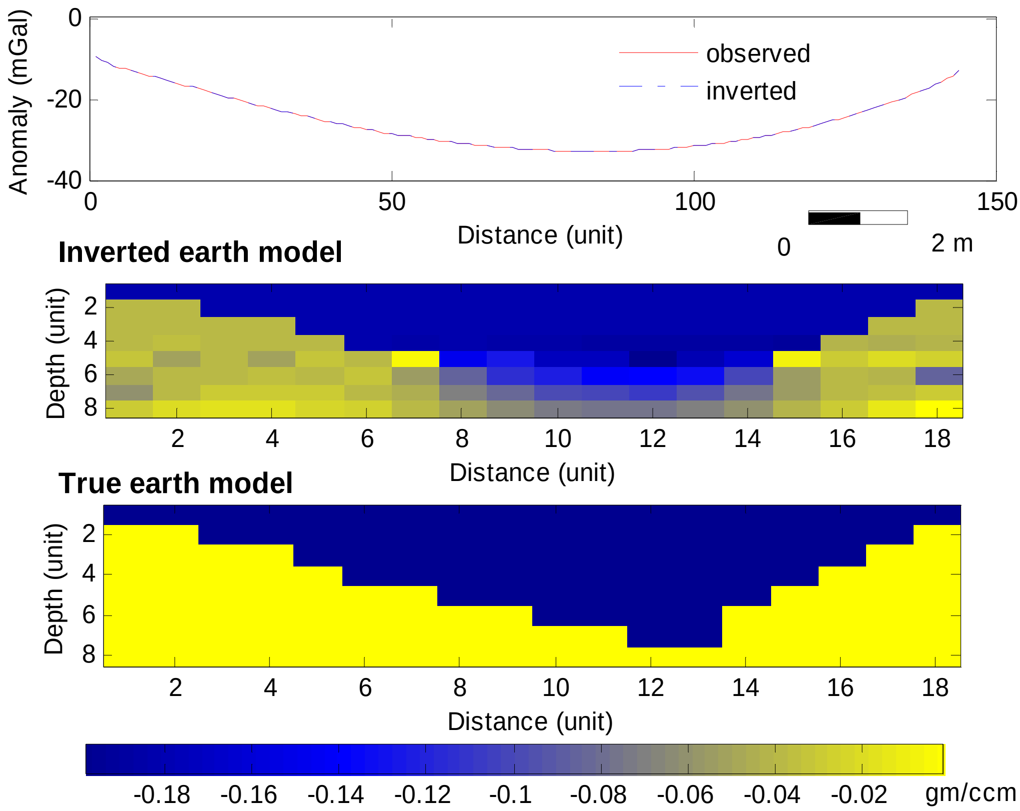

Figure 7. Inversion results of Model 2 (noise free).

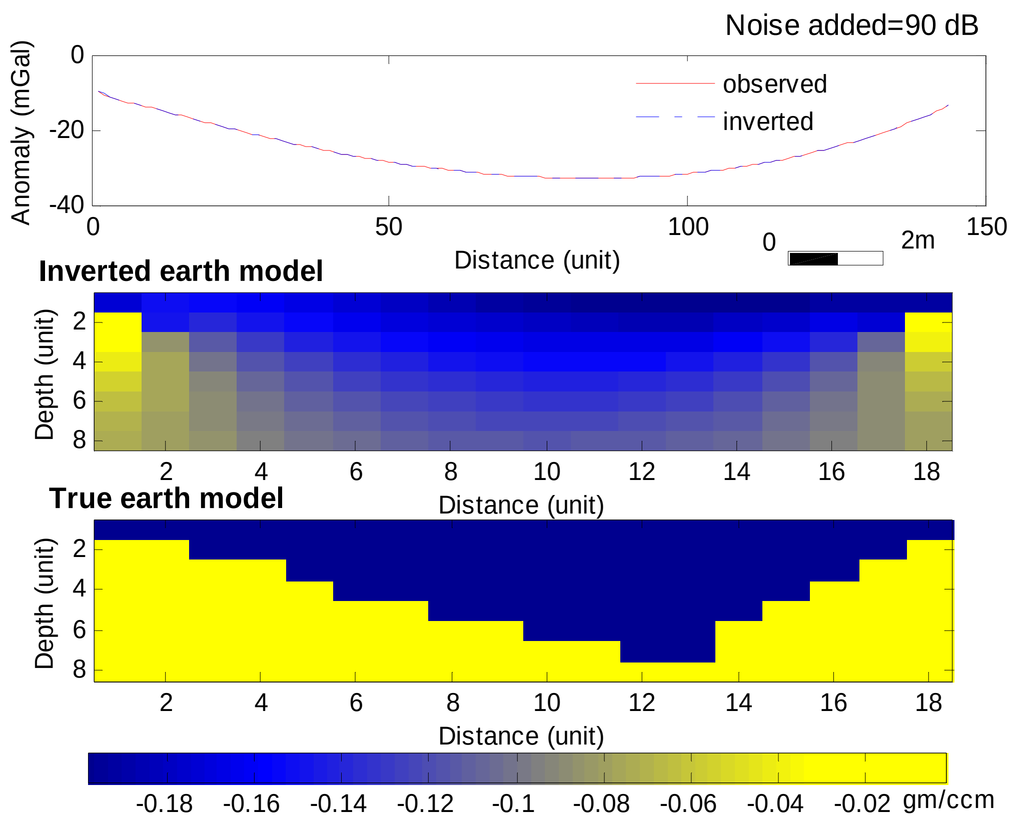

Figure 8. Inversion results of Model 2 after adding 90 DB noise.

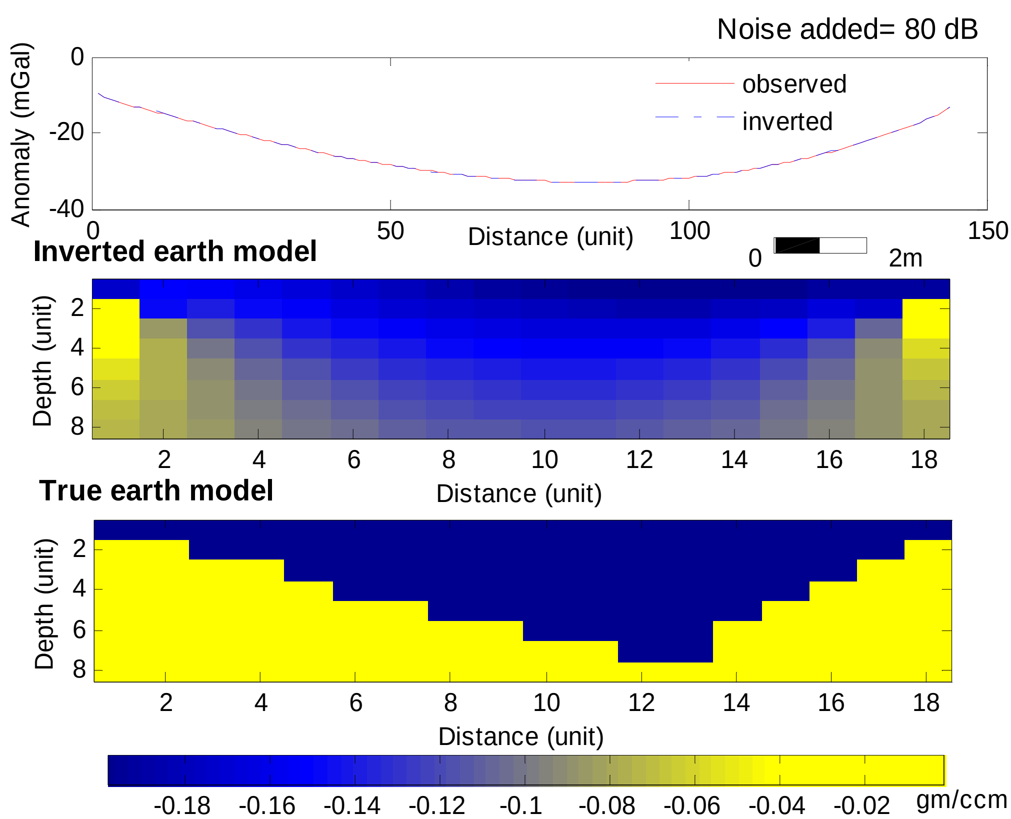

Figure 9. Inversion results of Model 2 after adding 80 DB noise.

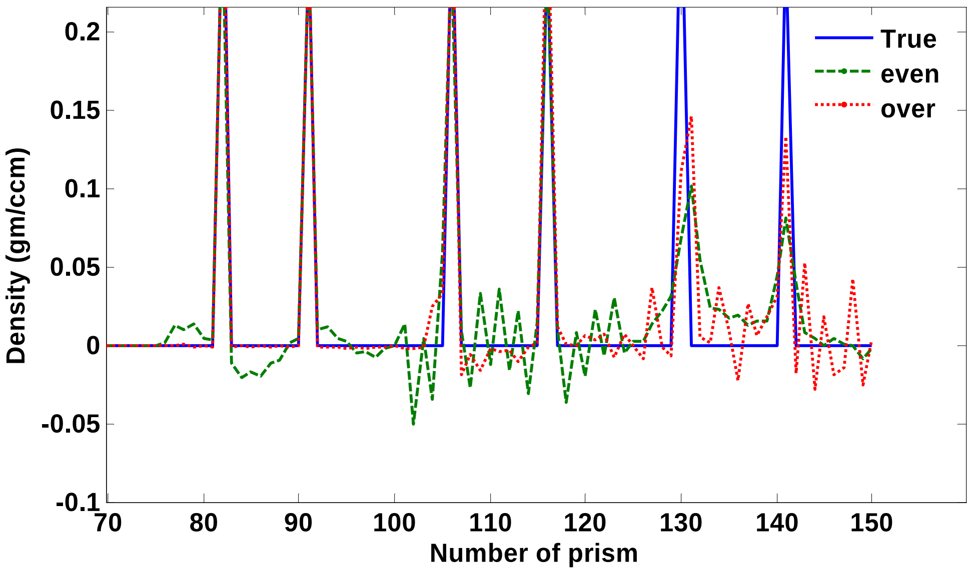

Figure 10. Comparison between evendetermined and overdetermined solutions for model 1 (using the same linear constraints).

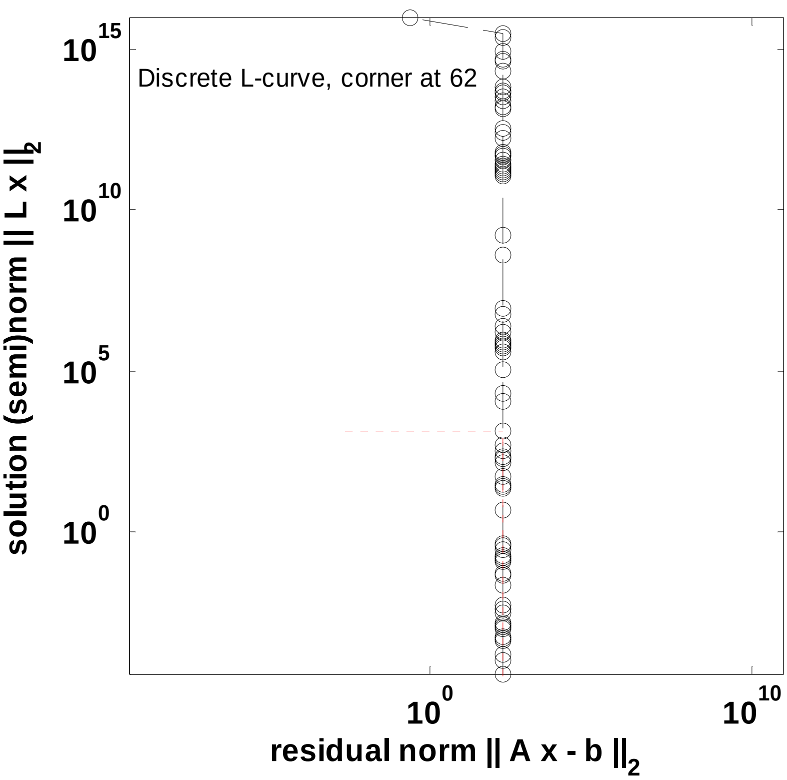

Figure 11. The L-curve for San Jacino Graben, southern California,USA.

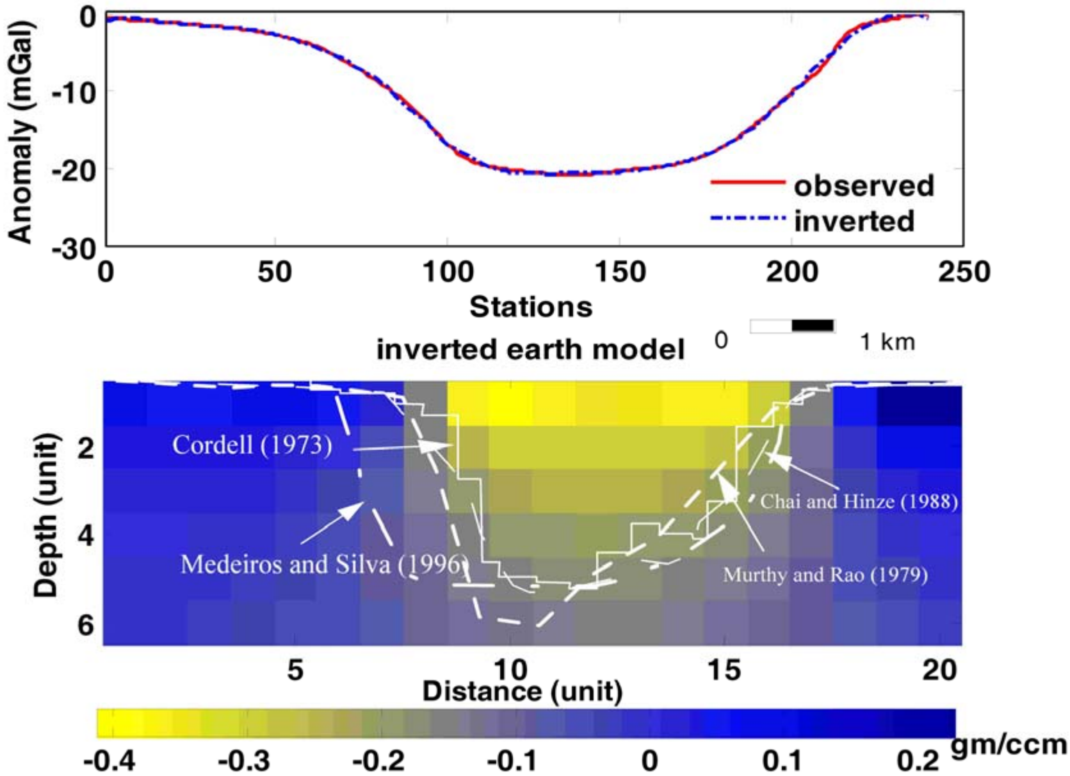

Figure 12. Inversion results of San Jacino, southern California, USA.